5-3 Dielectrics

Reading Assignment: *pp. 132-149*

Recall that if a **dielectric** material is immersed in an **electric field**, each atom/molecule in the material will form an **electric dipole!**

HO: The Polarization Vector

A. Polarization Charge Distribution

Q:

A: HO: Polarization Charge Distributions

B. Electric Flux Density

HO: Electric Flux Density

C. Field Equations in Dielectrics

Q:

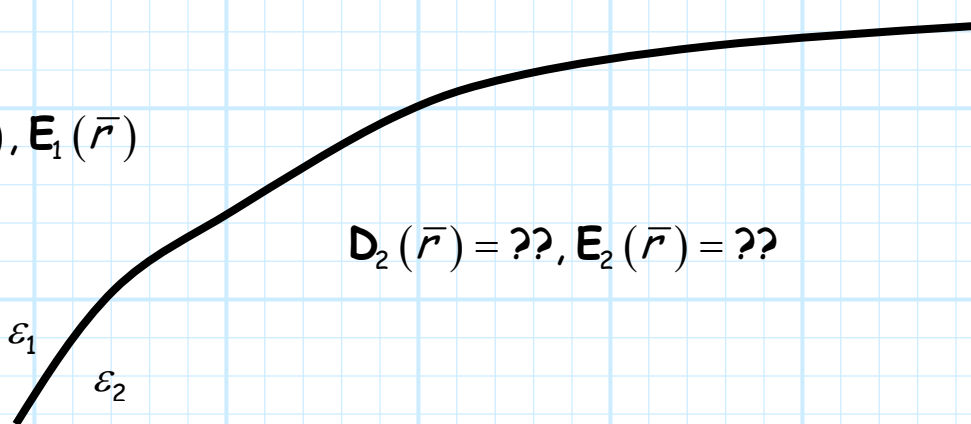A: HO: Electrostatic Field Equations in Dielectrics

D. Electric Boundary Conditions

Q:

$\mathbf{D}_1(\bar{r}), \mathbf{E}_1(\bar{r})$

$\mathbf{D}_2(\bar{r}) = ??, \mathbf{E}_2(\bar{r}) = ??$

ε_1

ε_2

A:

HO: Dielectric Boundary Conditions
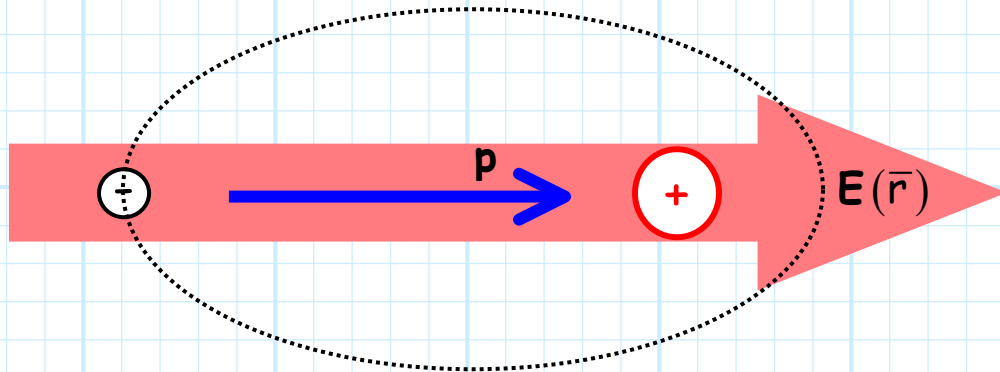
HO: Boundary Conditions on Perfect Conductors

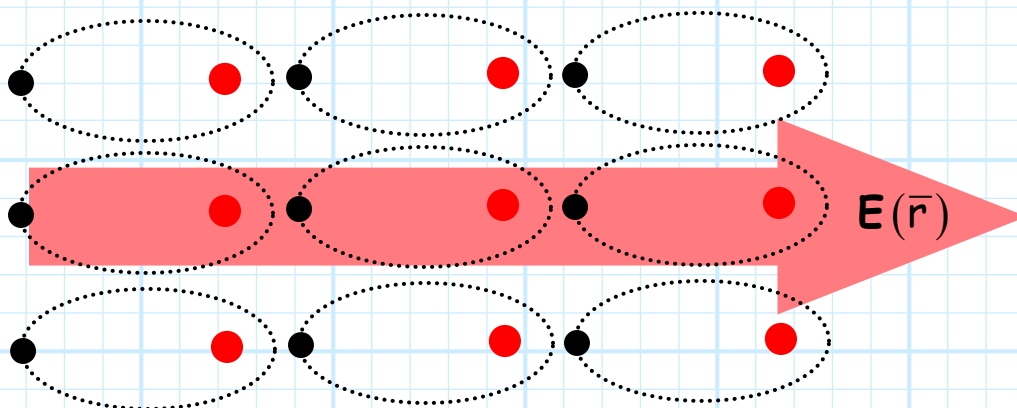**Example: Dielectric Boundary Conditions**

The Polarization Vector

Recall that in **dielectric materials** (i.e., insulators), the charges are **bound**.

As a result, atoms/molecules form **electric dipoles** when an electric field is present!

Note that even for some **small** volume Δv, there are **many** atoms/molecules present; therefore there will be **many** electric dipoles.

We will therefore define an **average** dipole moment, per unit volume, called the **Polarization Vector P** $(\bar{r})$.

$$P(\bar{r}) \doteq \frac{\sum p_n}{\Delta v} \qquad \left[\frac{\text{dipole moment}}{\text{unit volume}} = \frac{C}{m^2} \right]$$

where p_n is one of N dipole moments in volume Δv, centered at position $\bar{r}$. Note the polarization vector is a **vector field**. As a result, the direction and magnitude of the Polarization vector can change as function of position (i.e., a function of $\bar{r}$).

Q: *How are vector fields* **P** $(\bar{r})$ *and* **E** $(\bar{r})$ *related??*

A: Recall that the direction of each dipole moment is the same as the polarizing electric field. Thus **P** $(\bar{r})$ and **E** $(\bar{r})$ have the same direction. There magnitudes are related by a unitless scalar value $\chi_e(\bar{r})$, called **electric susceptibility**:

$$P(\bar{r}) = \varepsilon_0 \, \chi_e(\bar{r}) \, E(\bar{r})$$

Electric susceptibility is a **material parameter** indicating the "stretchability" of the dipoles.
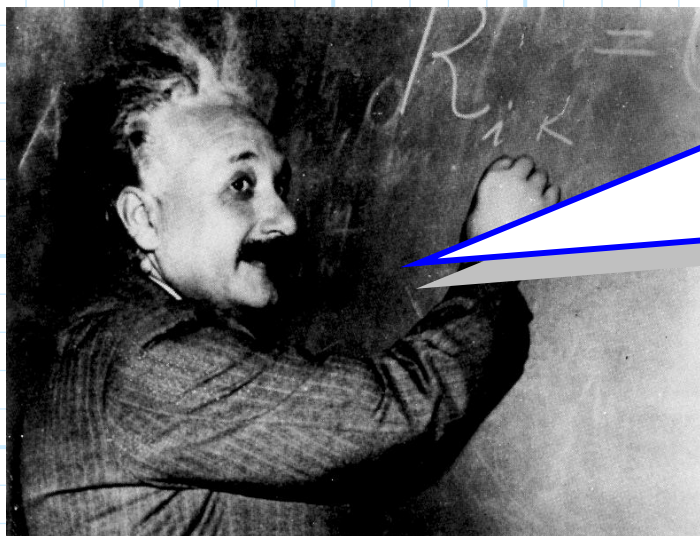
Q: *Can we determine the **fields** created by a polarized material?*

A: Recall the **electric potential field** created by **one** dipole is:

$$V(\overline{r}) = \frac{\mathbf{p} \cdot (\overline{r} - \overline{r}')}{4\pi\varepsilon_0 \left|\overline{r} - \overline{r}'\right|^3}$$

Therefore, using $d\,\mathbf{p} = \mathbf{P}(\overline{r})\,dv$, the electric potential field created by a **distribution of** dipoles (i.e., $\mathbf{P}(\overline{r})$) across some volume V is (see fig. 5.9):

$$V(\overline{r}) = \iiint\limits_V \frac{\mathbf{P}(\overline{r}') \cdot (\overline{r} - \overline{r}')}{4\pi\varepsilon_0 \left|\overline{r} - \overline{r}'\right|^3}\, dv'$$

Q: *But I thought **scalar** charge distributions $\rho_v(\overline{r})$ and $\rho_s(\overline{r})$ created the electric potential field $V(\overline{r})$. Now you are saying that electric fields are created by the **vector** field $\mathbf{P}(\overline{r})$!?!*

A: As we will soon see, the polarization **vector** $\mathbf{P}(\overline{r})$ creates equivalent charge **distributions**—we will get the correct answer for $V(\overline{r})$ from **either** source!

<u>Polarization Charge Distributions</u>

Consider a hunk of **dielectric** material with volume V.

Say this dielectric material is immersed in an **electric field** $\mathbf{E}(\bar{r})$, therefore creating atomic **dipoles** with density $\mathbf{P}(\bar{r})$.

Q: What **electric potential field** $V(\bar{r})$ is created by these diploes?

A: We know that:

$$V(\bar{r}) = \iiint_V \frac{\mathbf{P}(\bar{r}') \cdot (\bar{r}\text{-}\bar{r}')}{4\pi\varepsilon_0 |\bar{r}\text{-}\bar{r}'|^3} \, dv'$$

But, it can be shown that (p. 135):

$$V(\bar{r}) = \iiint_V \frac{\mathbf{P}(\bar{r}') \cdot (\bar{r}\text{-}\bar{r}')}{4\pi\varepsilon_0 |\bar{r}\text{-}\bar{r}'|^3} \, dv'$$

$$= \frac{1}{4\pi\varepsilon_0} \iiint_V \frac{-\nabla \cdot \mathbf{P}(\bar{r}')}{|\bar{r}\text{-}\bar{r}'|} \, dv' + \frac{1}{4\pi\varepsilon_0} \oiint_S \frac{\mathbf{P}(\bar{r}') \cdot \hat{a}_n(\bar{r})}{|\bar{r}\text{-}\bar{r}'|} \, ds'$$

where S is the **closed** surface that surrounds volume V, and $\hat{a}_n(\bar{r})$ is the unit vector **normal** to surface S (pointing **outward**).

This complicated result is only important when we compare it to the electric potential created by **volume** charge density $\rho_v(\bar{r})$ and **surface** charge density $\rho_s(\bar{r})$:

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_V \frac{\rho_v(\bar{r}')}{|\bar{r}-\bar{r}'|} \, dv'$$

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \iint_S \frac{\rho_s(\bar{r}')}{|\bar{r}-\bar{r}'|} \, ds'$$

If both volume and surface charge are present, the **total** electric potential field is:

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_V \frac{\rho_v(\bar{r}')}{|\bar{r}-\bar{r}'|} \, dv' + \frac{1}{4\pi\varepsilon_0} \iint_S \frac{\rho_s(\bar{r}')}{|\bar{r}-\bar{r}'|} \, ds'$$

Compare this expression to the previous integral involving the **polarization vector P**$(\bar{r})$. It is evident that the two expressions are equal if the following relations are true:

$$\rho_{vp}(\bar{r}) = -\nabla \cdot \mathbf{P}(\bar{r})$$

and

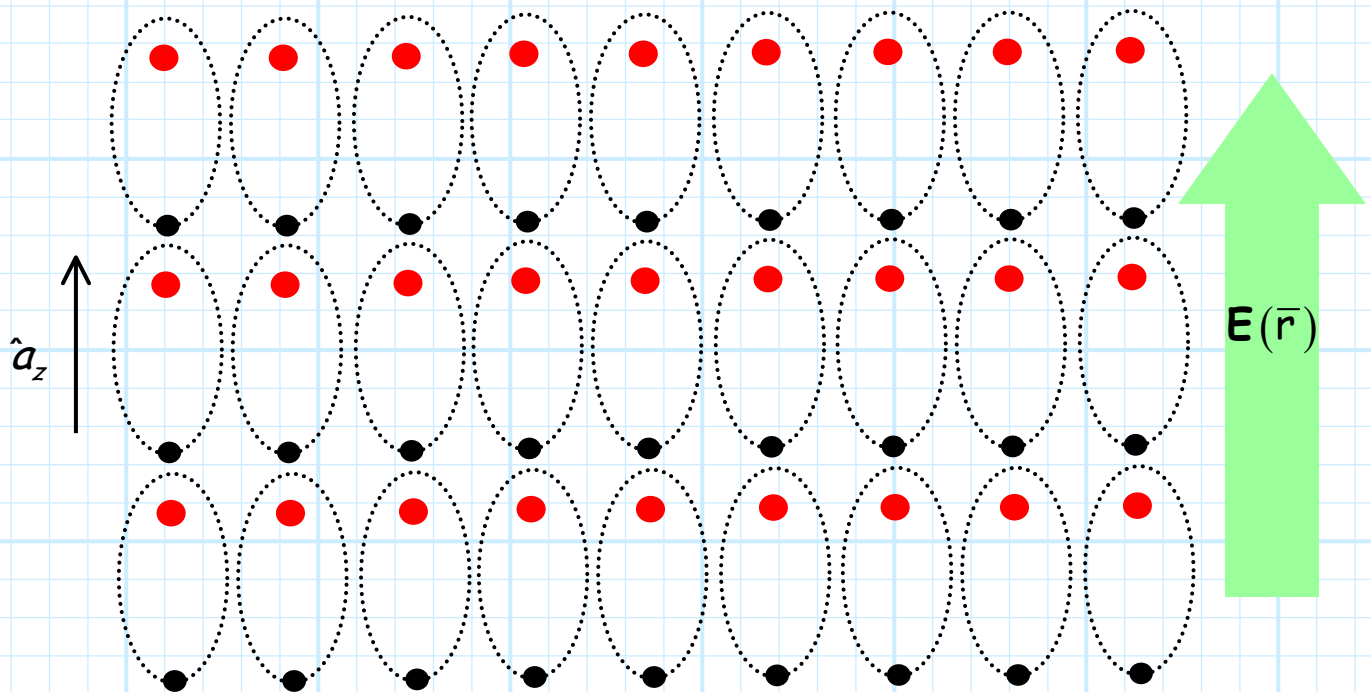$$\rho_{sp}(\bar{r}) = \mathbf{P}(\bar{r}) \cdot \hat{a}_n$$

The subscript p (e.g., ρ_{vp}, ρ_{sp}) indicates that these functions represent **equivalent charge densities** due to the due to the **dipoles** created in the dielectric.

In other words, the electric potential field $V(\bar{r})$ (and thus electric field $\mathbf{E}(\bar{r})$) created by the dipoles in the dielectric (i.e., $\mathbf{P}(\bar{r})$) is **indistinguishable** from the electric potential field created by the equivalent charge densities $\rho_{vp}(\bar{r})$ and $\rho_{sp}(\bar{r})$!

For example, consider a dielectric material immersed in an electric field, such that its polarization vector $\mathbf{P}(\bar{r})$ is:

$$\mathbf{P}(\bar{r}) = 3\,\hat{a}_z \quad \left[\frac{C}{m^2}\right]$$

Note since the polarization vector is a **constant**, the equivalent volume charge density is **zero**:

$$\rho_{vp}\left(\overline{r}\right) = -\nabla \cdot \mathbf{P}\left(\overline{r}\right)$$
$$= -\nabla \cdot 3\,\hat{a}_z$$
$$= 0$$

On the **top** surface of the dielectric ($\hat{a}_n = \hat{a}_z$), the equivalent **surface** charge is:

$$\rho_{sp}\left(\overline{r}\right) = \mathbf{P}\left(\overline{r}\right) \cdot \hat{a}_n$$
$$= 3\,\hat{a}_z \cdot \hat{a}_z$$
$$= 3 \qquad \left[\dfrac{C}{m^2}\right]$$

On the **bottom** of the dielectric ($\hat{a}_n = -\hat{a}_z$), the equivalent **surface** charge is:

$$\rho_{sp}\left(\overline{r}\right) = \mathbf{P}\left(\overline{r}\right) \cdot \hat{a}_n$$
$$= -3\,\hat{a}_z \cdot \hat{a}_z$$
$$= -3 \qquad \left[\dfrac{C}{m^2}\right]$$

On the **sides** of the dielectric material, the **surface** charge is **zero**, since $\hat{a}_z \cdot \hat{a}_n = 0$.

This result actually makes **physical** sense! Note at the **top** of dielectric, there is a layer of **positive** charge, and at the **bottom**, there is a layer of **negative** charge.

$$\hat{a}_z$$

$$\rho_{sp} = 3 \ C/m^2$$

$$\rho_{vp} = 0$$

$$\rho_{sp} = -3 \ C/m^2$$
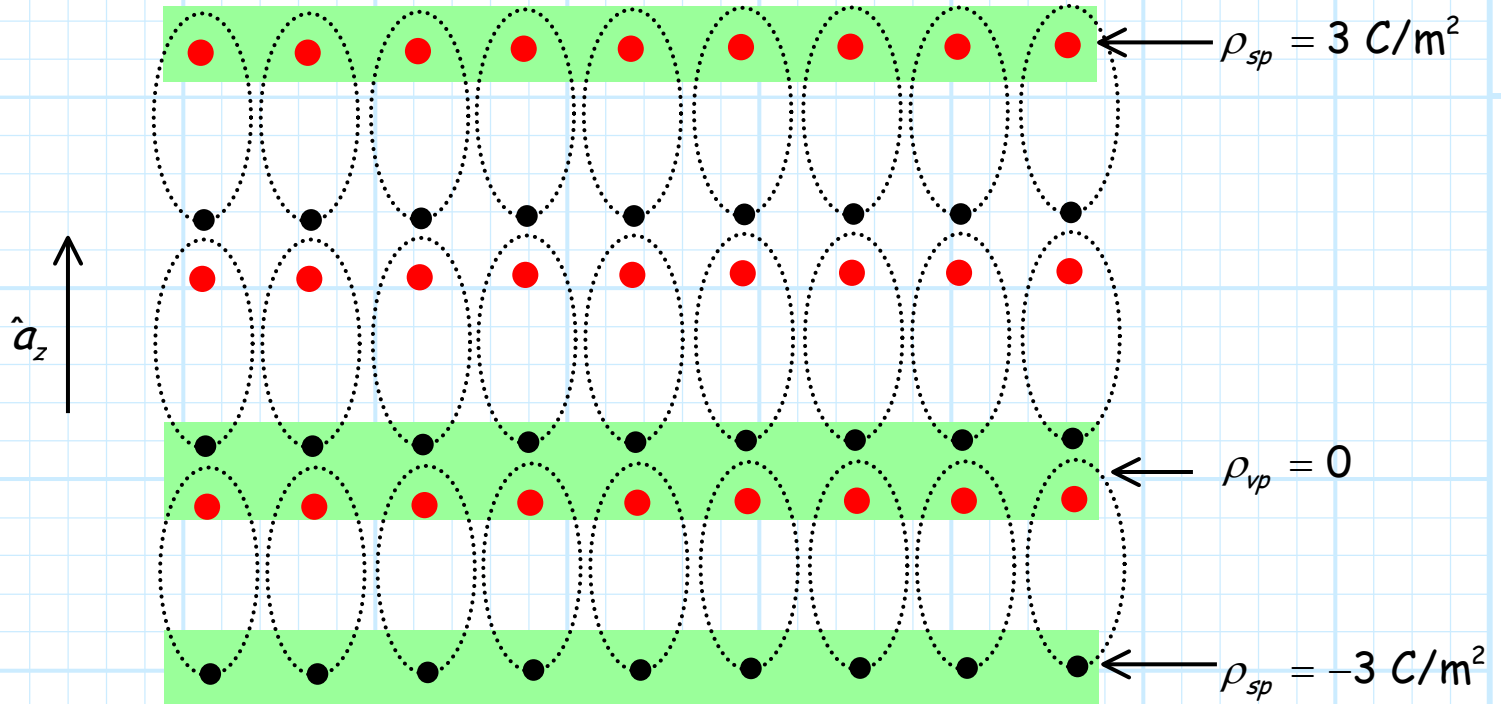
In the **middle** of the dielectric, there are **positive** charge layers on top of **negative** charge layers. The two add together and **cancel** each other, so that equivalent **volume** charge density is **zero.**

Finally, recall that there is no perfect dielectric, all materials will have some non-zero conductivity $\sigma(\bar{r})$.

As a result, we find that the total charge density within some material is the sum of the polarization charge density and the free charge (i.e., conducting charge) density:

$$\rho_{vT}\left(\overline{r}\right) = \rho_v\left(\overline{r}\right) + \rho_{vp}\left(\overline{r}\right)$$

Where:

$$\rho_{vT}\left(\overline{r}\right) \doteq \text{total charge density}$$

$$\rho_v\left(\overline{r}\right) \doteq \text{free charge density}$$

$$\rho_{vp}\left(\overline{r}\right) \doteq \text{polarization charge density}$$

This is likewise (as well as **more frequently!**) true for **surface** charge density:

$$\rho_{sT}\left(\overline{r}\right) = \rho_s\left(\overline{r}\right) + \rho_{sp}\left(\overline{r}\right)$$

Electric Flux Density

Yikes! Things have gotten **complicated**!

In free space, we found that charge $\rho_v(\bar{r})$ creates an electric field $\mathbf{E}(\bar{r})$.

Pretty **simple**! $\rho_v(\bar{r})$ ➡ $\mathbf{E}(\bar{r})$

But, if dielectric material is present, we find that charge $\rho_v(\bar{r})$ creates an **initial** electric field $\mathbf{E}_i(\bar{r})$. This electric field in turn **polarizes** the material, forming bound charge $\rho_{vp}(\bar{r})$. This bound charge, however, then creates its **own** electric field $\mathbf{E}_s(\bar{r})$ (sometimes called a **secondary** field), which modifies the initial electric field!

Not so simple! $\rho_v(\bar{r})$ ➡ $\mathbf{E}_i(\bar{r})$ ➡ $\rho_{vp}(\bar{r})$ ➡ $\mathbf{E}_s(\bar{r})$

The **total** electric field created by free charge when dielectric material is present is thus $\mathbf{E}(\bar{r}) = \mathbf{E}_i(\bar{r}) + \mathbf{E}_s(\bar{r})$.

> **Q:** *Isn't there some **easier** way to account for the effect of dielectric material??*

> **A:**   Yes there is! We use the concept of dielectric **permittivity**, and a new vector field called the **electric flux density** $\mathbf{D}(\bar{r})$.

To see how this works, first consider the point form of **Gauss's Law**:

$$\nabla \cdot \mathbf{E}(\bar{r}) = \frac{\rho_{vT}(\bar{r})}{\varepsilon_0}$$

where $\rho_{vT}(\bar{r})$ is the **total** charge density, consisting of both the **free** charge density $\rho_v(\bar{r})$ and **bound** charge density $\rho_{vp}(\bar{r})$:

$$\rho_{vT}(\bar{r}) = \rho_v(\bar{r}) + \rho_{vp}(\bar{r})$$

Therefore, we can write Gauss's Law as:

$$\varepsilon_0 \nabla \cdot \mathbf{E}(\bar{r}) = \rho_v(\bar{r}) + \rho_{vp}(\bar{r})$$

Recall the **bound** charge density is equal to:

$$\rho_{vp}(\bar{r}) = -\nabla \cdot \mathbf{P}(\bar{r})$$

Inserting into the above equation:

$$\varepsilon_0 \nabla \cdot \mathbf{E}(\bar{r}) = \rho_v(\bar{r}) - \nabla \cdot \mathbf{P}(\bar{r})$$

And rearranging:

$$\varepsilon_0 \nabla \cdot \mathbf{E}(\bar{r}) + \nabla \cdot \mathbf{P}(\bar{r}) = \rho_v(\bar{r})$$
$$\nabla \cdot \left[\varepsilon_0 \mathbf{E}(\bar{r}) + \mathbf{P}(\bar{r})\right] = \rho_v(\bar{r})$$

Note this final result says that the divergence of vector field $\varepsilon_0 \mathbf{E}(\bar{r}) + \mathbf{P}(\bar{r})$ is equal to the **free** charge density $\rho_v(\bar{r})$. Let's define this vector field the **electric flux density** $\mathbf{D}(\bar{r})$:

$$\text{electric flux density} \quad \mathbf{D}(\bar{r}) \doteq \varepsilon_0 \mathbf{E}(\bar{r}) + \mathbf{P}(\bar{r}) \quad \left[\frac{C}{m^2} \right]$$

Therefore, we can write a **new** form of Gauss's Law:

$$\nabla \cdot \mathbf{D}(\bar{r}) = \rho_v(\bar{r})$$

This equation says that the electric flux density $\mathbf{D}(\bar{r})$ **diverges** from **free** charge $\rho_v(\bar{r})$. In other words, the source of electric flux density is free charge $\rho_v(\bar{r})$--and free charge **only**!

* The electric field $\mathbf{E}(\bar{r})$ is created by **both** free charge and bound charge within the dielectric material.

* However, the electric flux density $\mathbf{D}(\bar{r})$ is created by **free** charge **only**—the bound charge within the dielectric material makes no difference with regard to $\mathbf{D}(\bar{r})$!

But wait! We can simplify this further. Recall that the polarization vector is related to electric field by susceptibility $\chi_e(\bar{r})$:

$$\mathbf{P}(\bar{r}) = \varepsilon_0 \chi_e(\bar{r}) \mathbf{E}(\bar{r})$$

Therefore the electric flux density is:

$$\mathbf{D}(\bar{r}) = \varepsilon_0 \mathbf{E}(\bar{r}) + \varepsilon_0 \chi_e(\bar{r}) \mathbf{E}(\bar{r})$$
$$= \varepsilon_0 \left(1 + \chi_e(\bar{r})\right) \mathbf{E}(\bar{r})$$

We can further simplify this by defining the **permittivity** of the medium (the dielectric material):

$$\text{permittivity } \varepsilon(\bar{r}) \doteq \varepsilon_0 \left(1 + \chi_e(\bar{r})\right)$$

And can further define **relative** permittivity:

$$\text{relative permittivity } \varepsilon_r(\bar{r}) \doteq \frac{\varepsilon(\bar{r})}{\varepsilon_0} = 1 + \chi_e(\bar{r})$$

Note therefore that $\varepsilon(\bar{r}) = \varepsilon_r(\bar{r})\,\varepsilon_0$.

We can thus write a **simple** relationship between electric flux density and electric field:

$$\mathbf{D}(\overline{r}) = \varepsilon(\overline{r})\mathbf{E}(\overline{r})$$
$$= \varepsilon_0\,\varepsilon_r(\overline{r})\mathbf{E}(\overline{r})$$

Like conductivity $\sigma(\overline{r})$, permittivity $\varepsilon(\overline{r})$ is a fundamental **material** parameter. Also like conductivity, it relates the electric field to another vector field.

Thus, we have an **alternative** way to view electrostatics:

1. Free charge $\rho_v(\overline{r})$ creates electric flux density $\mathbf{D}(\overline{r})$.

2. The electric field can be then determined by simply dividing $\mathbf{D}(\overline{r})$ by the material permittivity $\varepsilon(\overline{r})$ (i.e., $\mathbf{E}(\overline{r}) = \mathbf{D}(\overline{r})/\varepsilon(\overline{r})$).

$$\rho_v(\overline{r}) \implies \mathbf{D}(\overline{r}) \implies \mathbf{E}(\overline{r})$$

Electrostatic Field Equations in Dielectrics

The electrostatic equations for fields in **dielectric materials** are:

$$\nabla \times \mathbf{E}(\overline{r}) = 0$$

$$\nabla \cdot \mathbf{D}(\overline{r}) = \rho_v(\overline{r})$$

$$\mathbf{D}(\overline{r}) = \varepsilon(\overline{r})\mathbf{E}(\overline{r})$$

In **integral** form, these equations are:

$$\oint_C \mathbf{E}(\overline{r}) \cdot \overline{d\ell} = 0$$

$$\oiint_S \mathbf{D}(\overline{r}) \cdot \overline{ds} = Q_{enc}$$

$$\mathbf{D}(\overline{r}) = \varepsilon(\overline{r})\mathbf{E}(\overline{r})$$

Likewise, for free charge located in some **homogeneous** (i.e., constant) material with permittivity ε, we get the following relations:

$$\mathbf{E}(\bar{r}) = \frac{Q}{4\pi\varepsilon} \frac{\bar{r}\text{-}\bar{r}'}{|\bar{r}\text{-}\bar{r}'|} \quad \text{(for point charge Q)}$$

$$V(\bar{r}) = \frac{1}{4\pi\varepsilon} \iiint\limits_{V} \frac{\rho_v(\bar{r}')}{|\bar{r}\text{-}\bar{r}'|} \, dv'$$

$$\nabla^2 V(\bar{r}) = \frac{-\rho_v(\bar{r})}{\varepsilon}$$

In other words, for homogenous materials, **replace** ε_0 (the permittivity of free-space) with the more general permittivity value ε.
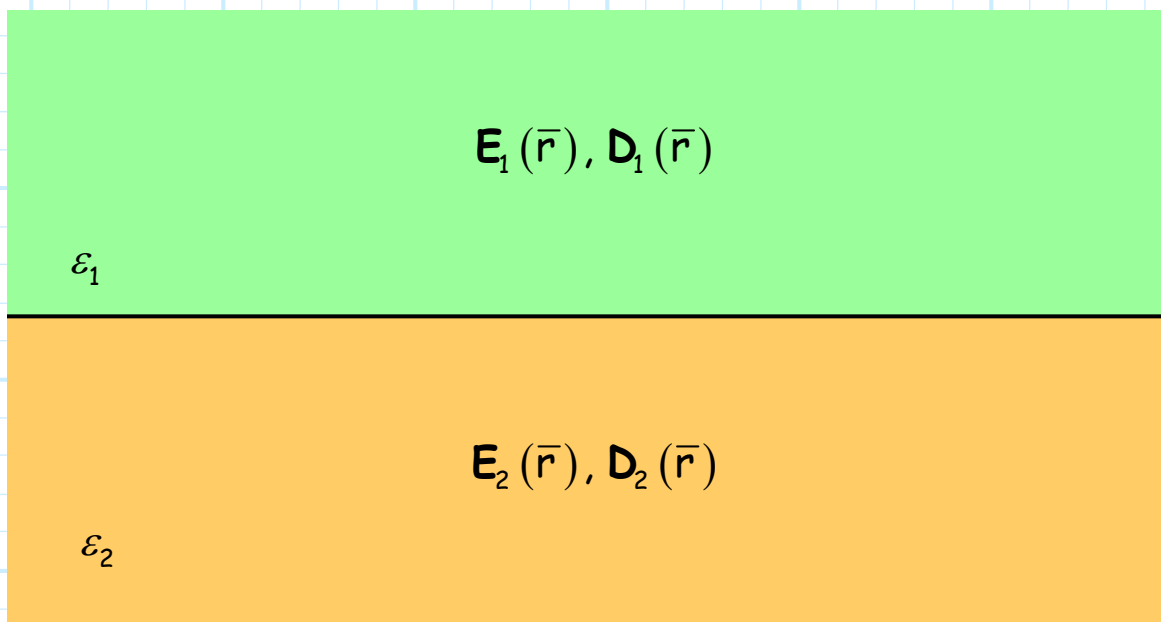
Pretty **simple** !

For **example**:

    If the media is **free-space**, use the permittivity of **free-space**.

    If the media is, for example, **plastic**, then use the permittivity of **plastic**.

Dielectric Boundary Conditions

Consider the **interface** between two dissimilar **dielectric** regions:

Say that an **electric field** is present in both regions, thus producing also an electric flux density ($\mathbf{D}(\bar{r}) = \varepsilon \mathbf{E}(\bar{r})$).

> **Q:** *How are the fields in dielectric* **region 1** *(i.e.,* $\mathbf{E}_1(\bar{r}), \mathbf{D}_1(\bar{r})$*) related to the fields in* **region 2** *(i.e.,* $\mathbf{E}_2(\bar{r}), \mathbf{D}_2(\bar{r})$*)?*

> **A:** They must satisfy the **dielectric boundary conditions** !

First, let's write the fields **at the dielectric interface** in terms of their **normal** ($E_n(\bar{r})$) and **tangential** ($E_t(\bar{r})$) vector components:
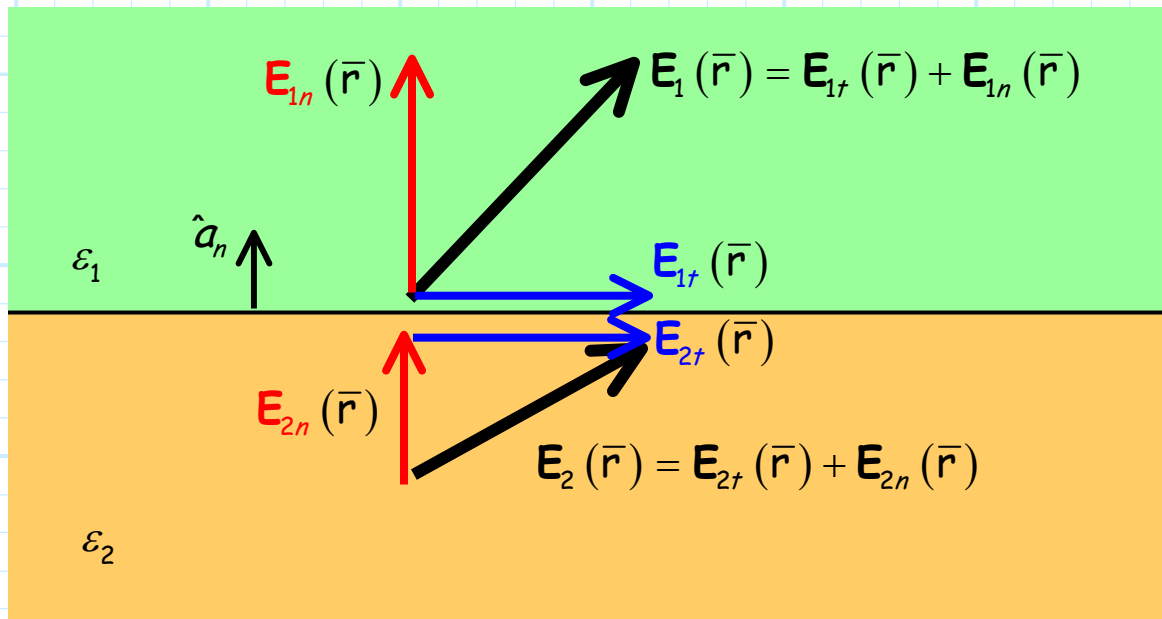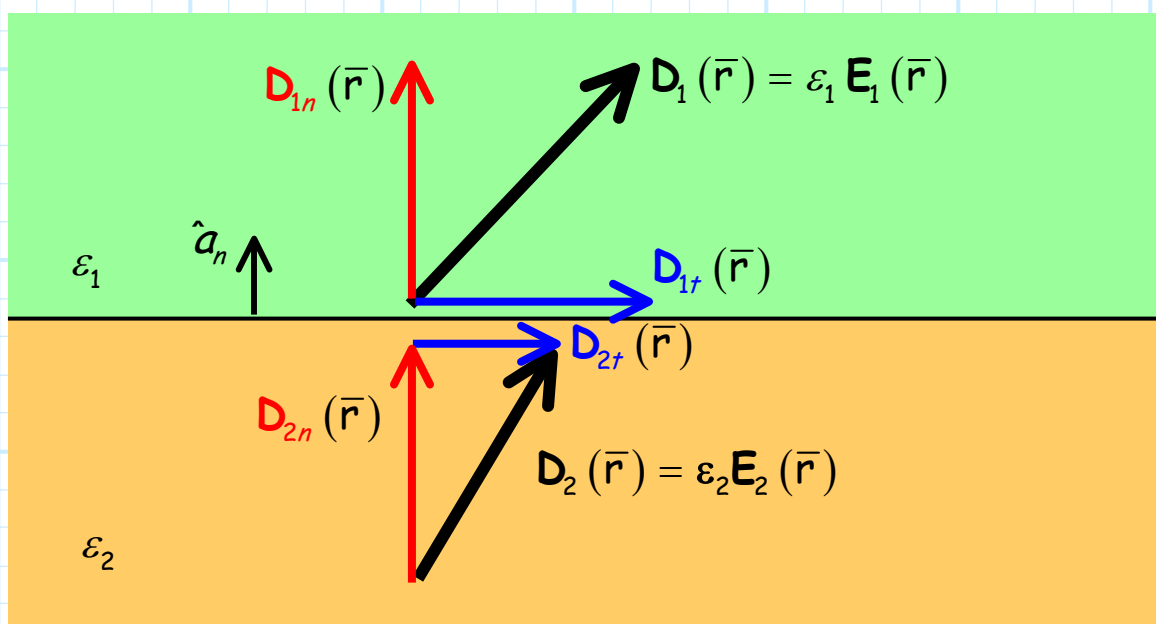

Our first boundary condition states that the **tangential** component of the electric field is **continuous** across a boundary. In other words:

$$\mathbf{E}_{1t}(\bar{r}_b) = \mathbf{E}_{2t}(\bar{r}_b)$$

where $\bar{r}_b$ denotes any point **on the boundary** (e.g., dielectric interface).

The **tangential** component of the electric field at **one** side of the dielectric boundary is **equal** to the tangential component at the **other** side !

We can likewise consider the **electric flux densities** on the dielectric interface in terms of their **normal** and **tangential** components:

The second dielectric boundary condition states that the **normal** vector component of the **electric flux density** is **continuous** across the dielectric boundary. In other words:

$$\mathbf{D}_{1n}(\bar{r}_b) = \mathbf{D}_{2n}(\bar{r}_b)$$

where $\bar{r}_b$ denotes any point **on the dielectric boundary** (i.e., dielectric interface).

Since $\mathbf{D}(\overline{r}) = \varepsilon\, \mathbf{E}(\overline{r})$, these boundary conditions can **likewise** be expressed as:

$$\mathbf{E}_{1t}(\overline{r}_b) = \mathbf{E}_{2t}(\overline{r}_b)$$

$$\frac{\mathbf{D}_{1t}(\overline{r}_b)}{\varepsilon_1} = \frac{\mathbf{D}_{2t}(\overline{r}_b)}{\varepsilon_2}$$

and as:

$$\mathbf{D}_{1n}(\overline{r}_b) = \mathbf{D}_{2n}(\overline{r}_b)$$
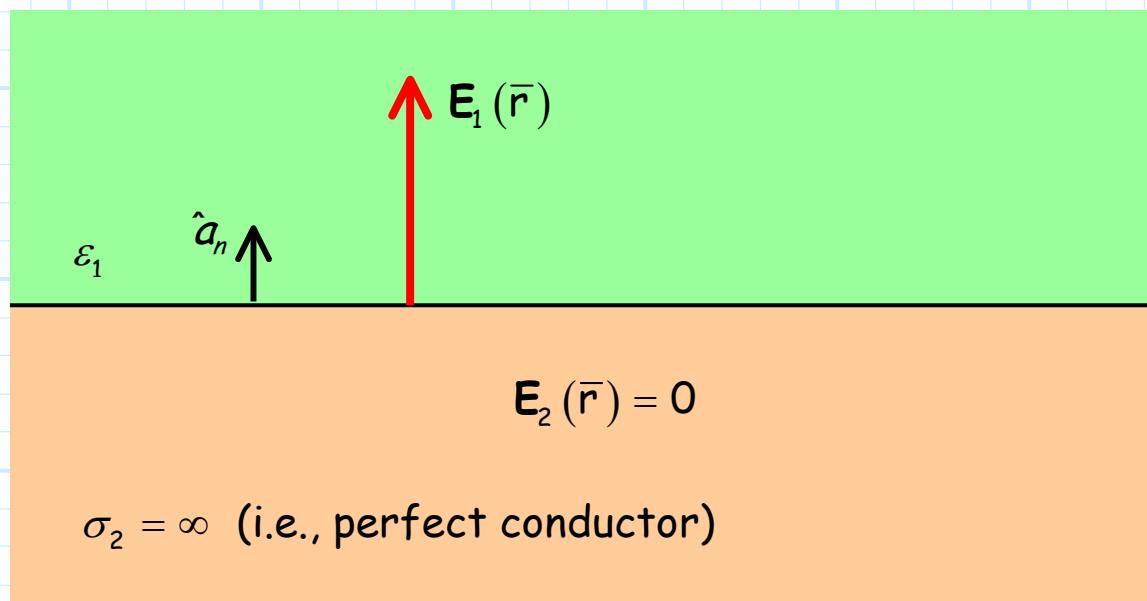
$$\varepsilon_1 \mathbf{E}_{1n}(\overline{r}_b) = \varepsilon_2 \mathbf{E}_{2n}(\overline{r}_b)$$

MAKE SURE <u>YOU</u> UNDERSTAND THIS:

These boundary conditions describe the relationships of the vector fields **at the dielectric interface** only (i.e., at points $\overline{r} = \overline{r}_b$)!!!! They say **nothing** about the value of the fields at points above or below the interface.

Boundary Conditions on Perfect Conductors

Consider the case where region 2 is a **perfect conductor**:

Recall $\mathbf{E}(\bar{r}) = 0$ in a perfect conductor. This of course means that **both** the tangential and normal component of $\mathbf{E}_2(\bar{r})$ are also equal to **zero**:

$$\mathbf{E}_{2t}(\bar{r}) = 0 = \mathbf{E}_{2n}(\bar{r})$$

And, since the **tangential** component of the electric field is **continuous** across the boundary, we find that **at the interface**:

$$\mathbf{E}_{1t}(\bar{r}_b) = \mathbf{E}_{2t}(\bar{r}_b) = 0$$

Think about what this means! The **tangential** vector component in the dielectric (at the dielectric/conductor boundary) is **zero**. Therefore, the electric field **at the boundary** only has a **normal** component:

$$\mathbf{E}_1(\bar{r}_b) = \mathbf{E}_{1n}(\bar{r}_b)$$

Therefore, we can say:

> The **electric field** on the **surface** of a **perfect conductor** is **orthogonal** (i.e., normal) to the conductor.

Q1: *What about the **electric flux density** $\mathbf{D}_1(\bar{r})$?*

A1: The relation $\mathbf{D}_1(\bar{r}) = \varepsilon_1 \mathbf{E}_1(\bar{r})$ is still of course valid, so that the **electric flux density** at the surface of the conductor must **also be orthogonal** to the conductor.

Q2: *But, we learned that the **normal** component of the **electric flux density** is **continuous** across an interface. If $\mathbf{D}_{2n}(\bar{r}) = 0$, why isn't $\mathbf{D}_{1n}(\bar{r}_b) = 0$?*

A2: Great question! The answer comes from a more **general** form of the **boundary condition.**

Consider again the interface of two **dissimilar dielectrics**. This time, however, there is some **surface charge distribution** $\rho_s(\bar{r}_b)$ (i.e., **free** charge!) **at the dielectric interface**:
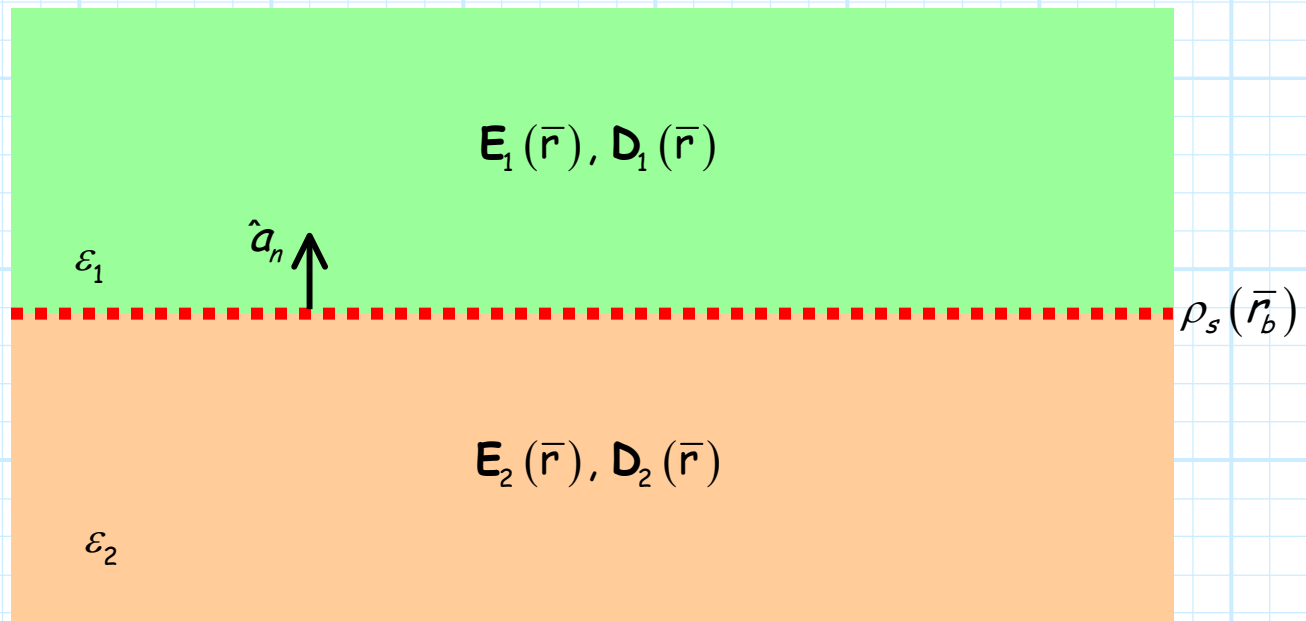
The **boundary condition** for this situation turns out to be:

$$\hat{a}_n \cdot \left[\mathbf{D}_{1n}(\bar{r}_b) - \mathbf{D}_{2n}(\bar{r}_b) \right] = \rho_s(\bar{r}_b)$$

$$D_{1n}(\bar{r}_b) - D_{2n}(\bar{r}_b) = \rho_s(\bar{r}_b)$$

where $D_n(\bar{r}_b) = \hat{a}_n \cdot \mathbf{D}_n(\bar{r}_b)$ is the **scalar component** of $\mathbf{D}_n(\bar{r}_b)$ (note the **units** of each side are C/m^2 !).

Note that **if** $\rho_s(\bar{r}_b) = 0$, this boundary condition returns (both physically and mathematically) to the case studied earlier—the **normal** component of the electric flux density is **continuous** across the interface.

This more **general** boundary condition is useful for the dielectric/**conductor** interface. Since $\mathbf{D}_2(\bar{r}) = 0$ in the conductor, we find that:

$$\hat{a}_n \cdot \left[\mathbf{D}_{1n}(\bar{r}_b) - \mathbf{D}_{2n}(\bar{r}_b) \right] = \rho_s(\bar{r}_b)$$

$$\hat{a}_n \cdot \mathbf{D}_{1n}(\bar{r}_b) = \rho_s(\bar{r}_b)$$

$$D_{1n}(\bar{r}_b) = \rho_s(\bar{r}_b)$$

In other words, the **normal** component of the **electric flux density** at the **conductor surface** is equal to the **charge density** on the conductor surface.

Note in a perfect conductor, there is **plenty** of **free** charge available to form this charge density ! Therefore, we find in **general** that $D_{1n} \neq 0$ at the surface of a conductor.

$$\mathbf{D}_1(\bar{r}_b)$$

ε_1 $\hat{a}_n$

$$\rho_s(\bar{r}_b)$$

$$\mathbf{D}_2(\bar{r}) = 0$$

$\sigma_2 = \infty$ (i.e., perfect conductor)
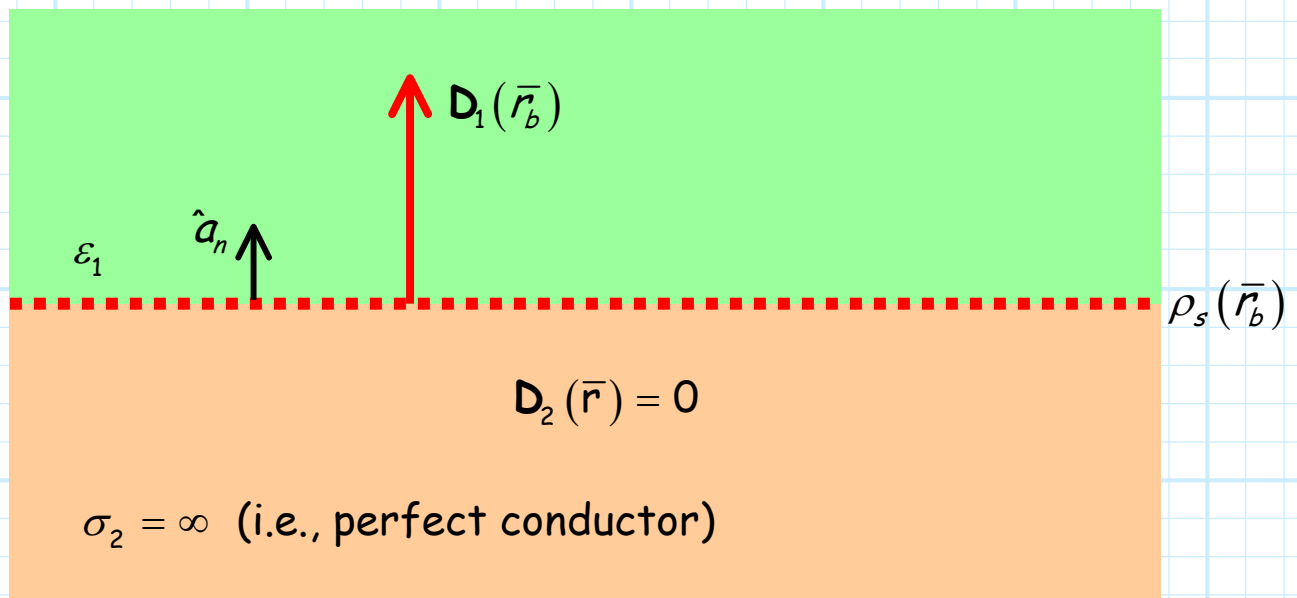
Summarizing, the boundary conditions for the **tangential** components field components at a **dielectric/conductor** interface are:

$$\mathbf{E}_{1t}\left(\bar{r}_b\right) = 0$$

$$\mathbf{D}_{1t}\left(\bar{r}_b\right) = 0$$

but for the **normal** field components:

$$D_{1n}\left(\bar{r}_b\right) = \rho_s\left(\bar{r}_b\right)$$

$$E_{1n}\left(\bar{r}_b\right) = \frac{\rho_s\left(\bar{r}_b\right)}{\varepsilon_1}$$

Again, these boundary conditions describe the fields **at the conductor/dielectric interface.** They say **nothing** about the value of the fields at locations above this interface.
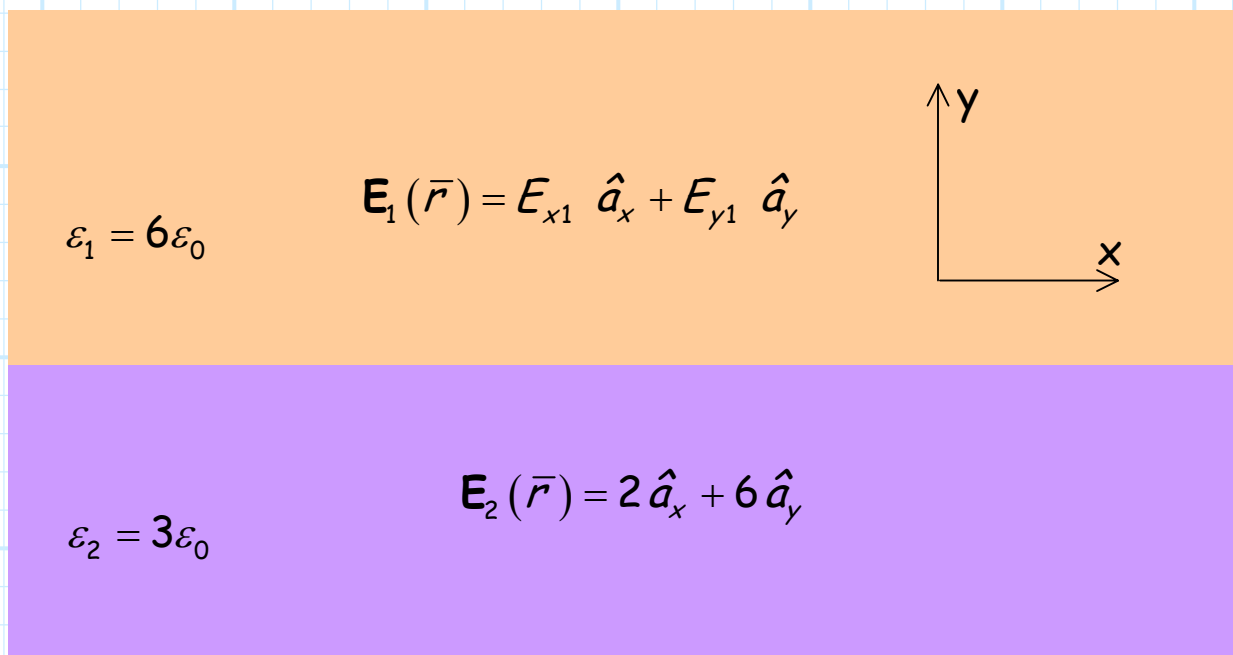
Example: Boundary Conditions

Two slabs of dissimilar **dielectric** material share a common **boundary**, as shown below.

It is known that the electric field in the **lower** dielectric region is:

$$\mathbf{E}_2(\bar{r}) = 2\,\hat{a}_x + 6\,\hat{a}_y \quad \left[\frac{V}{m}\right]$$

and it is known that the electric field in the top region is likewise some **constant** field:

$$\mathbf{E}_1(\bar{r}) = E_{x1}\ \hat{a}_x + E_{y1}\ \hat{a}_y \quad \left[\frac{V}{m}\right]$$

$$\mathbf{E}_1(\bar{r}) = E_{x1}\ \hat{a}_x + E_{y1}\ \hat{a}_y$$

$\varepsilon_1 = 6\varepsilon_0$

$$\mathbf{E}_2(\bar{r}) = 2\,\hat{a}_x + 6\,\hat{a}_y$$

$\varepsilon_2 = 3\varepsilon_0$

In **each** dielectric region, let's determine (in terms of ε_0):

     1) the **electric field**
     2) the **electric flux density**
     3) the **bound** *volume* **charge density** (i.e., the equivalent polarization charge density) within the dielectric.
     4) the **bound** *surface* **charge density** (i.e., the equivalent polarization charge density) at the dielectric interface

Since we already know the electric field in the region, let's evaluate **region 2** first.

We can easily determine the **electric flux density** within the region:

$$\mathbf{D}_2\left(\bar{r}\right) = \varepsilon_2 \mathbf{E}_2\left(\bar{r}\right)$$
$$= 3\,\varepsilon_0\left(2\,\hat{a}_x + 6\,\hat{a}_y\right)$$
$$= 6\varepsilon_0\,\hat{a}_x + 18\varepsilon_0\,\hat{a}_y \quad \left[\frac{C}{m^2}\right]$$

Likewise, the polarization vector within the region is:

$$\mathbf{P}_2\left(\bar{r}\right) = \varepsilon_0 \chi_{e2}\,\mathbf{E}_2\left(\bar{r}\right)$$
$$= \varepsilon_0\left(\varepsilon_{r2} - 1\right)\left(2\,\hat{a}_x + 6\,\hat{a}_y\right)$$
$$= \varepsilon_0\left(3 - 1\right)\left(2\,\hat{a}_x + 6\,\hat{a}_y\right)$$
$$= 4\varepsilon_0\,\hat{a}_x + 12\varepsilon_0\,\hat{a}_y \quad \left[\frac{C}{m^2}\right]$$

Q: *Why did we determine the **polarization** vector? It is **not** one of the quantities this problem asked for!*

A: True! But the problem **did** ask for the equivalent **bound charge densities** (both volume and surface) within the dielectric. We need to know polarization vector $\mathbf{P}(\bar{r})$ to find this **bound** charge!

Recall the bound **volume** charge density is:

$$\rho_{vp}(\bar{r}) = -\nabla \cdot \mathbf{P}(\bar{r})$$
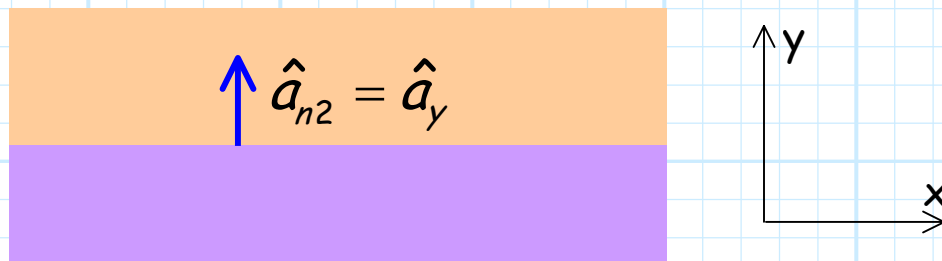
and the bound **surface** charge density is:

$$\rho_{sp}(\bar{r}) = \mathbf{P}(\bar{r}) \cdot \hat{a}_n$$

Since the polarization vector $\mathbf{P}(\bar{r})$ is a **constant** (i.e., it has precisely the same magnitude and direction at every point with region 2), we find that the divergence of $\mathbf{P}(\bar{r})$ is **zero,** and thus the volume bound charge density is zero within the region:

$$\rho_{vp2}(\bar{r}) = -\nabla \cdot \mathbf{P}_2(\bar{r})$$

$$= -\nabla \cdot \left(4\varepsilon_0 \, \hat{a}_x + 12\varepsilon_0 \, \hat{a}_y \right)$$

$$= 0 \quad \left[C\!\!\Big/_{m^3} \right]$$

However, we find that the **surface** bound charge density is **not** zero!

Note that the unit vector normal to the **surface** of the bottom dielectric slab is $\hat{a}_{n2} = \hat{a}_y$:

Since the polarization vector is constant, we know that its value at the **dielectric interface** is likewise equal to $4\varepsilon_0\,\hat{a}_x + 12\varepsilon_0\,\hat{a}_y$.

Thus, the equivalent polarization (i.e., **bound**) **surface charge density** on the top of region 2 (at the dielectric interface) is

$$\rho_{sp2}\left(\bar{r}_b\right) = \mathbf{P}_2\left(\bar{r}_b\right) \cdot \hat{a}_{n2}$$
$$= \left(4\varepsilon_0\,\hat{a}_x + 12\varepsilon_0\,\hat{a}_y\right) \cdot \hat{a}_y$$
$$= 12\varepsilon_0 \quad \left[\frac{C}{m^2}\right]$$

Now, let's determine these same quantities for **region 1** (i.e., the **top** dielectric slab).

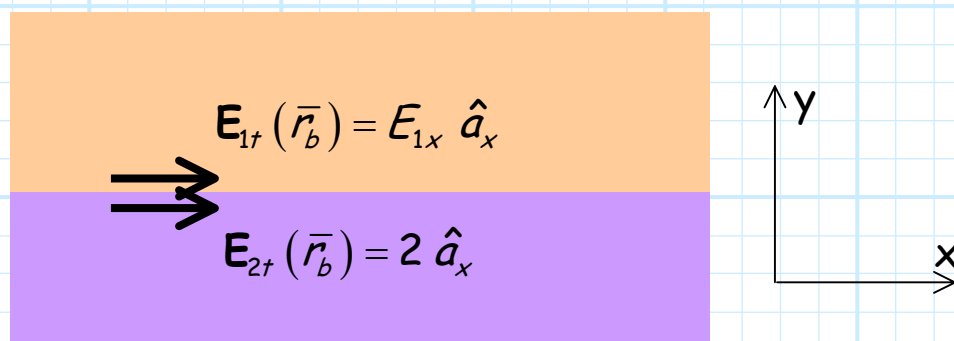**Q1:** *How the heck can we do this? We don't know* **anything** *about the fields in region 1 !*

A1: True! We don't know $\mathbf{E}_1\left(\bar{r}\right)$ or $\mathbf{D}_1\left(\bar{r}\right)$ or even $\mathbf{P}_1\left(\bar{r}\right)$. However, we know the **next** best thing—we know $\mathbf{E}_2\left(\bar{r}\right)$ and $\mathbf{D}_2\left(\bar{r}\right)$ and even $\mathbf{P}_2\left(\bar{r}\right)$!

Q2: *Huh!?!*

A2: We can use **boundary conditions** to transfer our solutions from region 2 into region 1!

First, we note that **at the dielectric interface**, the vector components of the electric fields **tangential** to the interface are $\mathbf{E}_{1t}(\bar{r}_b) = E_{1x}\,\hat{a}_x$ and $\mathbf{E}_{2t}(\bar{r}_b) = 2\,\hat{a}_x$:

$$\mathbf{E}_{1t}(\bar{r}_b) = E_{1x}\,\hat{a}_x$$

$$\mathbf{E}_{2t}(\bar{r}_b) = 2\,\hat{a}_x$$

$\uparrow$ y

$\rightarrow$ x

Thus, applying the **boundary condition** $\mathbf{E}_{1t}(\bar{r}_b) = \mathbf{E}_{2t}(\bar{r}_b)$, we find:

$$\mathbf{E}_{1t}(\bar{r}_b) = \mathbf{E}_{2t}(\bar{r}_b)$$
$$E_{1x}\,\hat{a}_x = 2\,\hat{a}_x$$
$$E_{1x}\,\hat{a}_x \cdot \hat{a}_x = 2\,\hat{a}_x \cdot \hat{a}_x$$
$$E_{1x} = 2$$

Likewise, we note that **at the dielectric interface**, the vector components of the electric fields **normal** to the interface are $\mathbf{E}_{1n}(\bar{r}_b) = E_{1y}\,\hat{a}_y$ and $\mathbf{E}_{2n}(\bar{r}_b) = 6\,\hat{a}_y$:
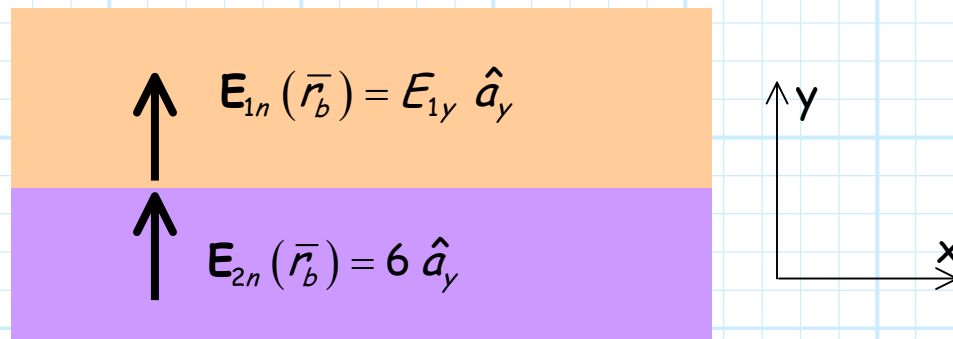
Here, we can apply a **second boundary condition**, $\varepsilon_1 \mathbf{E}_{1n}(\bar{r}_b) = \varepsilon_2 \mathbf{E}_{2n}(\bar{r}_b)$:

$$\varepsilon_1 \mathbf{E}_{1n}(\bar{r}_b) = \varepsilon_2 \mathbf{E}_{2n}(\bar{r}_b)$$
$$6\varepsilon_0 \, E_{y1} \, \hat{a}_y = 3\varepsilon_0 \, 6 \, \hat{a}_y$$
$$E_{y1} \, \hat{a}_y = 3 \, \hat{a}_y$$
$$E_{y1} \, \hat{a}_y \cdot \hat{a}_y = 3 \, \hat{a}_y \cdot \hat{a}_y$$
$$E_{y1} = 3$$

Thus, we have concluded using boundary conditions that $E_{x1} = 2$ and $E_{y1} = 3$, or the electric field in the top region is:

$$\mathbf{E}_1(\bar{r}) = 2 \, \hat{a}_x + 3 \, \hat{a}_y \quad \left[V/m\right]$$

Likewise, we can find the **electric flux density** by multiplying by the permittivity of region 1 ($\varepsilon_1 = 6\varepsilon_0$):

$$\mathbf{D}_1(\bar{r}) = \varepsilon_1 \mathbf{E}_1(\bar{r})$$
$$= 12 \, \varepsilon_0 \, \hat{a}_x + 18 \, \varepsilon_0 \, \hat{a}_y \quad \left[C/m^2\right]$$

Note we could have solved this problem **another** way!

Instead of applying boundary conditions to $\mathbf{E}_2(\bar{r})$, we **could** have applied them to **electric flux density** $\mathbf{D}_2(\bar{r})$:

$$\mathbf{D}_2(\bar{r}) = 6\varepsilon_0\,\hat{a}_x + 18\varepsilon_0\,\hat{a}_y \quad \left[\frac{C}{m^2}\right]$$

We know that the **electric flux density** within region 1 must be constant, i.e.:

$$\mathbf{D}_1(\bar{r}) = D_{x1}\,\hat{a}_x + D_{y1}\,\hat{a}_y \quad \left[\frac{C}{m^2}\right]$$

and that the vector fields $\mathbf{D}_1(\bar{r})$ and $\mathbf{D}_2(\bar{r})$ **at the interface** are related by the **boundary conditions**:

$$\frac{\mathbf{D}_{1t}(\bar{r}_b)}{\varepsilon_1} = \frac{\mathbf{D}_{2t}(\bar{r}_b)}{\varepsilon_2}$$

and

$$\mathbf{D}_{1n}(\bar{r}_b) = \mathbf{D}_{2n}(\bar{r}_b)$$

It is evident that for this problem:

$$\mathbf{D}_{1t}(\bar{r}_b) = D_{x1}\,\hat{a}_x$$

$$\mathbf{D}_{1n}(\bar{r}_b) = D_{y1}\,\hat{a}_y$$

and for region 2:

$$\mathbf{D}_{2t}\left(\overline{r}_b\right) = 12\varepsilon_0\,\hat{a}_x$$

$$\mathbf{D}_{2n}\left(\overline{r}_b\right) = 18\varepsilon_0\,\hat{a}_y$$

Combining the results, we find the **two boundary conditions** are:

$$\frac{\mathbf{D}_{1t}\left(\overline{r}_b\right)}{\varepsilon_1} = \frac{\mathbf{D}_{2t}\left(\overline{r}_b\right)}{\varepsilon_2}$$

$$\frac{D_{1x}\,\hat{a}_x}{6\varepsilon_0} = \frac{6\varepsilon_0\hat{a}_x}{3\varepsilon_0}$$

$$D_{1x}\,\hat{a}_x = 12\varepsilon_0\,\hat{a}_x$$

$$D_{1x}\,\hat{a}_x \cdot \hat{a}_x = 12\varepsilon_0\,\hat{a}_x \cdot \hat{a}_x$$

$$D_{1x} = 12\varepsilon_0$$

and:

$$\mathbf{D}_{1n}\left(\overline{r}_b\right) = \mathbf{D}_{2n}\left(\overline{r}_b\right)$$

$$D_{1y}\,\hat{a}_y = 18\,\varepsilon_0\,\hat{a}_y$$

$$D_{1y}\,\hat{a}_y \cdot \hat{a}_y = 18\,\varepsilon_0\,\hat{a}_y \cdot \hat{a}_y$$

$$D_{1y} = 18\,\varepsilon_0$$

Therefore, we find that the **electric flux density** is:

$$\mathbf{D}_1\left(\overline{r}\right) = 12\varepsilon_0\,\hat{a}_x + 18\varepsilon_0\,\hat{a}_y \quad \left[C\!\!\diagup\!\!m^2\right]$$

Precisely the **same** result as before!

Likewise, we can find the **electric field** in region 1 by dividing by the dielectric permittivity:

$$\mathbf{E}_1(\overline{r}) = \frac{\mathbf{D}_1(\overline{r})}{\varepsilon_1}$$

$$= \frac{12\varepsilon_0 \, \hat{a}_x + 18\varepsilon_0 \, \hat{a}_y}{6\varepsilon_0}$$

$$= 2 \, \hat{a}_x + 3 \, \hat{a}_y \quad \left[\frac{V}{m} \right]$$

Again, the **same** result as before!

Now, finishing this problem, we need to find the **polarization** vector $\mathbf{P}_1(\overline{r})$:

$$\mathbf{P}_1(\overline{r}) = \varepsilon_0 (\varepsilon_{r1} - 1) \mathbf{E}_1(\overline{r})$$

$$= \varepsilon_0 (6 - 1)(2 \, \hat{a}_x + 3 \, \hat{a}_y)$$

$$= 10 \, \varepsilon_0 \, \hat{a}_x + 15 \, \varepsilon_0 \, \hat{a}_y \quad \left[\frac{C}{m^2} \right]$$

Thus, the **volume** charge density of **bound** charge is again **zero**:

$$\rho_{vp1}(\overline{r}) = -\nabla \cdot \mathbf{P}_1(\overline{r})$$

$$= -\nabla \cdot (10 \, \varepsilon_0 \, \hat{a}_x + 15 \, \varepsilon_0 \, \hat{a}_y)$$

$$= 0$$

However, we again find that the **surface** bound charge density is **not** zero!

Note that the unit vector **normal** to the **bottom** surface of the **top** dielectric slab points **downward**, i.e., $\hat{a}_{n1} = -\hat{a}_y$:
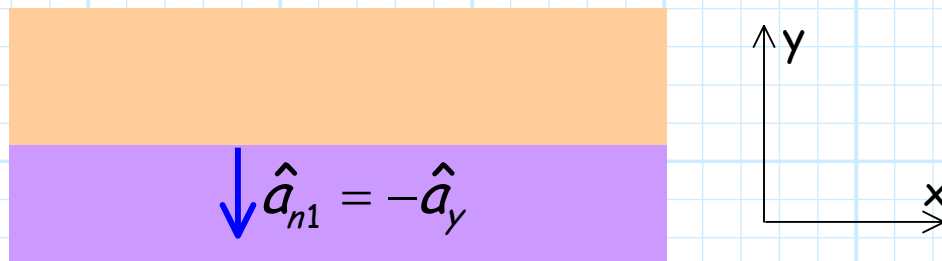
Since the polarization vector is **constant**, we know that its value **at the dielectric interface** is likewise equal to $10\varepsilon_0 \hat{a}_x + 15\varepsilon_0 \hat{a}_y$.

Thus, the equivalent polarization (i.e., **bound**) **surface charge density** on the bottom of region 1 (at the dielectric interface) is:

$$\rho_{sp1}(\bar{r}_b) = \mathbf{P}_1(\bar{r}_b) \cdot \hat{a}_{n1}$$

$$= \left(10\varepsilon_0 \hat{a}_x + 15\varepsilon_0 \hat{a}_y\right) \cdot \left(-\hat{a}_y\right)$$

$$= -15\varepsilon_0 \quad \left[\frac{C}{m^2}\right]$$

Now, we can determine the **net** surface charge density of **bound** charge that is lying **on the dielectric interface**:

$$\rho_{sp}(\bar{r}_b) = \rho_{sp1}(\bar{r}_b) + \rho_{sp2}(\bar{r}_b)$$

$$= -15\varepsilon_0 + 12\varepsilon_0$$

$$= -3\varepsilon_0 \quad \left[\frac{C}{m^2}\right]$$